Indian Statistical Institute, Bangalore Centre

B.Math (Hons.) III Year, First Semester

Backpaper Examination - 2013-2014

Complex Analysis

Time: 3 hours | January 7, 2014 | Instructor: Bhaskar Bagchi

Full Marks : 100.

1. (a) If a holomorphic function is injective on its domain then show that it can't have any essential singularity.

   (b) If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an injective meromorphic function then show that $f(z) \equiv \frac{az+l}{cz+d}$ for some constants a, l, c, d.

   [10+10=20]

2. (a) Define the residue of a holomorphic function at an isolated singularity.

   (b) State and prove the residue theorem.

   [5+15=20]

3. (a) If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function then show that for each $z \in \Omega$ there is a positive integer m_z such that f is m_z - to - one on some neighborhood of z.

   (b) Prove that we can't have $m_z = 2$ for all $z \in \Omega$.

   [12+8=20]

4. Use the fundamental theorem of Gauss to compute $\int\limits_{-\infty}^{\infty} e^{itx}\, e^{-x^2/2} dx$ for $t \in \mathbb{R}$. (You may use without proof the fact that for $t = 0$, the value of this integral is $\sqrt{2\pi}$.)

   [20]

5. Let $\Omega = \{x + iy : x > 1, y \in \mathbb{R}\}$. Define $\zeta : \Omega \to \mathbb{C}$ by $\zeta(z) = \sum\limits_{n=1}^{\infty} n^{-z}$.

   (a) Prove that this series converges locally uniformly on Ω. Hence deduce that ζ is holomorphic on Ω.

   (b) Prove that $\zeta(z) \neq 0$ for z in Ω.

   [12+8=20]